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The swirling round laminar jet

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Abstract. The swirling round laminar jet in an unbounded viscous fluid is investigated in this paper. The axisymmetric laminar jet with a swirling velocity is simulated by a linear-momentum source and an angular-momentum source, both located at the origin. The first-order and the second-order solutions in the far field have been obtained by solving the complete Navier–Stokes equations. It is found that the first-order solution is the well-known round-laminar-jet solution without the swirling velocity obtained by Landau [2] and Squire [3]. The second-order solution represents a pure rotating flow. The swirling velocity predicted by the present solution is compared with that obtained by Loitsyanskii [15] and Görtler [16], who solved the corresponding boundary-layer equations. It is found that the swirling velocity predicted by the present theory is smaller than that obtained from the boundary-layer equations.

1. Introduction

Solutions for laminar and turbulent jets have important applications in aerodynamics, combustion technology, and many other fields. The recent development in submerged jets of liquid metal combustion (see Hughes et al. [1]) may lead to the production of new propulsion systems for underwater vehicles. In order to maximize the combustion efficiency, the liquid jet is given a swirling velocity when leaving the nozzle orifice. However, even for laminar flows, there are only a few exact or asymptotic solutions of the Navier–Stokes equations due to mathematical difficulties.

The steady-state flow field of an incompressible fluid in an unbounded space due to a point source of momentum was studied by Landau [2] and Squire [3] independently, and it is called the Landau–Squire round laminar jet, which is one of the few exact solutions of nonlinear axisymmetric Navier–Stokes equations. Schlichting [4] solved the same problem using the boundary-layer equations at large Reynolds numbers in 1933. The development of a round laminar jet produced by a point force in an unbounded fluid was analyzed by Sozou and Pickering [5]. In their paper, the similarity method was applied and then the Navier–Stokes equations were solved numerically. Sozou [6] obtained an analytical solution for the initial stage of the development of a round laminar jet by solving the unsteady Stokes equations instead of the full Navier–Stokes equations.

Taylor [7] and Kraemer [8] studied the flow induced by a slender jet, with or without walls, by an inviscid potential-flow theory as a first approximation. On the other hand, exact solutions for a laminar, axisymmetric jet emerging from an orifice in a plane or conical wall were given by Squire [9] and Morgan [10], respectively. However, their solutions cannot satisfy the no-slip boundary conditions. Potsch [11] reconsidered and generalized the exact

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solutions for axisymmetric laminar jets from an orifice in a conical wall with neat solution forms. Both Morgan and Potsch stated that a non-trivial similarity solution for the jet flow satisfying the no-slip condition at conical walls did not exist. Squire and Potsch also showed that, in the presence of walls, the induced flow is not an inviscid flow, even in the limiting case of very high orifice Reynolds numbers. Schneider [12] found a non-trivial similarity solution satisfying the no-slip condition at the wall, by using the matched asymptotic-expansion method. In Schneider's solution, the outer solution (induced flow) was matched to the inner solution (jet). With respect to the outer flow region, the slender jet acted as a line mass sink of constant strength. He concluded that a non-trivial similarity solution, for the axisymmetric flow induced by a jet satisfying the no-slip condition at walls, exists in the limiting case as the Reynolds number approaches infinity. Schneider [13] showed that, at a finite distance, depending on the orifice Reynolds number, the jet diameter becomes very large, and a recirculating flow was predicted by using a multiple-scaling approach. Zauner [14] confirmed the existence of a recirculating flow by experiments.

The viscous swirling jet, in which the interaction between the azimuthal and the axial velocity components is important, is less understood even without walls. An asymptotic solution for a swirling jet was obtained by Loitsyanskii [15] and Görtler [16] applying the boundary-layer equations at large Reynolds numbers in the far field from the jet source. The higher-order terms of the asymptotic expansion were derived by Fal'kovich [17] in order to analyze the stream with more swirl, as well as the effect of twisting on the axial velocity profile. Zubtsov [18] analyzed a weak swirling jet, using a self-similar method to construct an asymptotic solution of the boundary-layer equations with small circulation. The only attempt to solve the swirling axisymmetric laminar jet with the Navier–Stokes equations was made by Wygnanski [19]. He assumed that there was a similarity solution with no-slip boundary conditions on the wall, although this assumption was not correct since no similarity solutions existed with the presence of the wall at finite Reynolds numbers.

Long [20] obtained a solution for a line vortex in an infinite viscous fluid. Similarity arguments led to a reduction of the Navier–Stokes equations to a set of ordinary differential equations, which can be simplified by a boundary-layer approximation. The simplified equations were integrated numerically by Long [20]. The inviscid stability of Long's vortex was studied by Foster and Duck [21] and by Foster and Smith [22], the latter included many references on stability problems involving a variety of vortices. However, the stability problem will not be studied in the present paper.

Yih et al. [23] found a class of exact solutions of the Navier–Stokes equations for conical vortices. On the conical surface, the slip condition was used. The azimuthal velocity was assumed to be inversely proportional to the distance. Thus the angular momentum of the entire fluid was either infinite or zero.

In the present paper we shall investigate the swirling round laminar jet, due to a point linear-momentum source and a point angular-momentum source of finite magnitude, in an unbounded viscous fluid by obtaining an asymptotic solution of the Navier–Stokes equations at a large distance from the linear- and angular-momentum sources. Various limiting forms of our solution are discussed and compared with other approximate solutions.

2. Governing equations and boundary conditions

Let us consider the motion of an incompressible fluid produced by a point force and a point torque at the origin of a spherical polar coordinate system (r, θ, ϕ). The direction of the force and the torque is along the axis $\theta = 0$. Therefore, the flow field is axially symmetric about this axis. The velocity vector is denoted by $\mathbf{u} = (u, v, w)$, where u, v, and w are the velocity components in the r, θ, and ϕ directions, respectively. Because of the axial symmetry, the velocity $\mathbf{u}$ is independent of ϕ and the continuity equation becomes

$$\frac{1}{r^2}\frac{\partial(r^2 u)}{\partial r} + \frac{1}{r\sin\theta}\frac{\partial(v\sin\theta)}{\partial\theta} = 0\,. \tag{1}$$

It is convenient to introduce a Stokes stream function ψ for an axisymmetric flow such that

$$u = \frac{1}{r^2\sin\theta}\frac{\partial\psi}{\partial\theta}\,, \tag{2a}$$

$$v = -\frac{1}{r\sin\theta}\frac{\partial\psi}{\partial r}\,. \tag{2b}$$

For a steady flow, the governing equations are the axisymmetric Navier–Stokes equations with body forces neglected,

$$\nabla\left(\frac{\mathbf{u}\cdot\mathbf{u}}{2}\right) - \mathbf{u}\times(\nabla\times\mathbf{u}) = -\frac{\nabla p}{\rho} - \nu\,\nabla\times(\nabla\times\mathbf{u})\,, \tag{3}$$

where ρ is the density, p the pressure, and ν the kinematic viscosity coefficient. Let r be the radius of a sphere centered at the origin and S be its surface. The integration of the momentum-flux tensor on the spherical surface gives the point force $\mathbf{F}$ exerted by the sphere on the surrounding fluid, and the integration of the angular momentum flux tensor gives the point torque $\mathbf{M}$. $\mathbf{F}$ and $\mathbf{M}$ act along the axis, $\theta = 0$. Hence

$$\mathbf{F} = \int_S (\mathbf{\Pi}\cdot\mathbf{n})\,\mathrm{d}S\,, \tag{4a}$$

$$\mathbf{M} = \int_S \mathbf{r}\times(\mathbf{\Pi}\cdot\mathbf{n})\,\mathrm{d}S\,, \tag{4b}$$

where $\mathbf{\Pi}$ is the momentum-flux tensor, $\boldsymbol{r}$ the position vector, and $\boldsymbol{n}$ the unit normal vector pointing away from the origin. In a Cartesian coordinate system, the momentum-flux tensor can be expressed as

$$\Pi_{ij} = \rho u_i u_j + p\delta_{ij} - \mu(u_{i,j} + u_{j,i})\,, \tag{5}$$

where $i, j = 1, 2, 3$, δ_{ij} is the Kronecker delta and μ is the dynamic viscosity coefficient. The boundary conditions at infinity are

$$\mathbf{u} = 0,\ p = 0 \quad \text{as } r \to \infty\,. \tag{6}$$

The value of the Stokes streamfunction ψ is assumed to be zero at $\theta = 0$.

Since the given force and torque are in the same direction, we can reduce (4a) and (4b) from vector forms to scalar forms by the following procedure. The ϕ-component of $\mathbf{\Pi}\cdot\mathbf{n}$ should vanish after the integration due to the axial symmetry. The resultant force $\mathbf{F}$ is in the direction of $\theta = 0$ with a magnitude

$$F = 2\pi \int_0^{\pi} \left\{ \left(\rho u^2 + p - 2\mu \frac{\partial u}{\partial r} \right) \cos\theta - \left[\rho uv - \mu r \frac{\partial}{\partial r}\left(\frac{v}{r}\right) - \frac{\mu}{r}\frac{\partial u}{\partial \theta} \right] \sin\theta \right\} r^2 \sin\theta \, d\theta \,. \tag{7}$$

Similarly, the r-component of $\mathbf{\Pi}\cdot\mathbf{n}$ vanishes after taking a cross product with $\mathbf{r}$, and the θ-component of $\mathbf{\Pi}\cdot\mathbf{n}$ vanishes after the integration due to the axial symmetry. Hence the resultant torque is also in the direction of $\theta = 0$ with a magnitude

$$M = 2\pi \int_0^{\pi} \left[\rho uw - \mu r \frac{\partial}{\partial r}\left(\frac{w}{r}\right) \right] r^3 \sin^2\theta \, d\theta \,. \tag{8}$$

3. Asymptotic solutions

Because the velocity and the pressure vanish at infinity, we propose to expand the velocity and the pressure as power series in $1/r$ in the far field with the coefficients functions of θ only. Thus, the stream function ψ, the ϕ-component velocity, and the pressure are assumed to be

$$\psi = \nu \left[ar + \sum_{i=0}^{\infty} \frac{a_i}{r^i} \right], \tag{9}$$

$$w = \nu \sum_{i=1}^{\infty} \frac{b_i}{r^i}, \tag{10}$$

$$p = \rho \nu^2 \sum_{i=1}^{\infty} \frac{c_i}{r^i}, \tag{11}$$

where a, a_0, a_i, b_i and c_i $(i = 1, 2, 3, \ldots)$ are functions of θ only. This technique is similar to the boundary-layer solution for the case with swirl by Loitsyanskii [15]. Substituting ψ into (2a) and (2b), we have

$$u = \frac{\nu}{\sin\theta} \left[\frac{a'}{r} + \sum_{i=0}^{\infty} \frac{a_i'}{r^{i+2}} \right], \tag{12}$$

$$v = -\frac{\nu}{\sin\theta} \left[\frac{a}{r} - \sum_{i=1}^{\infty} \frac{i a_i}{r^{i+2}} \right], \tag{13}$$

where $'$ denotes the derivative with respect to θ. Substituting the velocity $\mathbf{u}$ and the pressure p into the governing equations and boundary conditions, and collecting the coefficients of like terms of $1/r$, we have a series of ordinary differential equations and corresponding boundary conditions.

If we examine the order of each term in the Navier–Stokes equations by assuming that the velocity is of the order of r^n, where n is a negative integer, we would find that the terms on the left-hand-side of the Navier–Stokes equations are of the order of r^{2n-1}, and the order of the terms on the right-hand-side equation is r^{n-2}. When $n < -1$, the order of the inertial terms (left-hand-side equation) is always lower than that of the viscous terms. This makes all differential equations linear when $n < -1$, as the nonlinear inertial terms become known functions. When $n = -1$, the Navier–Stokes equations are of the order of r^{-3} on both sides, and the solution is the well-known solution for a round laminar jet. The expansion of the Navier–Stokes equations in r^{-1} and r^{-2}, for the pressure term only, shows that $c_1 = 0$. The expansion of the Navier–Stokes equations in r^{-3} are

$$\frac{a'''}{\sin\theta} + \frac{a''}{\sin^2\theta}(a - \cos\theta) + \frac{a'}{\sin^3\theta}(1 + a'\sin\theta - a\cos\theta) + \frac{a^2}{\sin^2\theta} = -2c_2 - b_1^2\,, \tag{14}$$

$$\frac{a''}{\sin\theta} - \frac{a'}{\sin^2\theta}(\cos\theta + a) + a^2\frac{\cos\theta}{\sin^3\theta} = c_2' - b_1^2\cot\theta\,, \tag{15}$$

$$b_1'' + b_1'\left(\cot\theta + \frac{a}{\sin\theta}\right) + \frac{b_1}{\sin^2\theta}(a\cos\theta - 1) = 0\,, \tag{16}$$

where Equations (14), (15), and (16) correspond to the r, θ, and ϕ components of the Navier–Stokes equations. The corresponding force condition in the order of r^0 and the torque condition in the order of r are

$$F = 2\pi\rho\nu^2 \int_0^\pi [(a')^2\cot\theta + a'\cos\theta + aa' + (2a + a'')\sin\theta + c_2\sin\theta\cos\theta]\,d\theta\,, \tag{17}$$

$$0 = \int_0^\pi b_1(a' + 2\sin\theta)\sin\theta\,d\theta\,. \tag{18}$$

The solution of the above differential-equation set is that for a round laminar jet (see Landau [2], Squire [3], and Batchelor [24]):

$$a(\theta) = \frac{2\sin^2\theta}{1 + c - \cos\theta}\,, \tag{19}$$

$$c_2(\theta) = \frac{4[(c+1)\cos\theta - 1]}{(c + 1 - \cos\theta)^2}\,, \tag{20}$$

$$b_1(\theta) = 0\,, \tag{21}$$

$$\frac{F}{2\pi\rho\nu^2} = \frac{32(c+1)}{3c(c+2)} + 4(c+1)^2\ln\left(\frac{c}{c+2}\right) + 8(c+1)\,, \tag{22}$$

where c is a positive constant. As pointed out by Batchelor [24], the condition that the flow is to be free from singularities on the axis of symmetry (i.e. at $\theta = 0$ and $\theta = \pi$) except at $r = 0$ was applied in obtaining the above Landau–Squire round-laminar-jet solution. A more precise "jet condition" in deriving this classical solution was given by Geurst [25]. As F approaches infinity, c approaches zero. As F approaches zero, c approaches infinity. The dimensionless force parameter $F/2\pi\rho\nu^2$ is plotted in Fig. 1 versus the value of c.

The expansion of the governing equation in r^{-4} yields

$$\frac{a_0'''}{\sin\theta} + \frac{a_0''}{\sin^2\theta}(a - \cos\theta) + \frac{a_0'}{\sin^3\theta}(1 - a\cos\theta + 3a'\sin\theta) = -3c_3 - 2b_1b_2\,, \tag{23}$$

$$\frac{2a_0''}{\sin\theta} - 2a_0'\frac{\cos\theta}{\sin^2\theta} = c_3' - 2b_1b_2\cot\theta\,, \tag{24}$$

$$b_2'' + b_2'\left(\cot\theta + \frac{a}{\sin\theta}\right) + \frac{b_2}{\sin^2\theta}(2\sin^2\theta - 1 + a'\sin\theta + a\cos\theta) = 0\,, \tag{25}$$

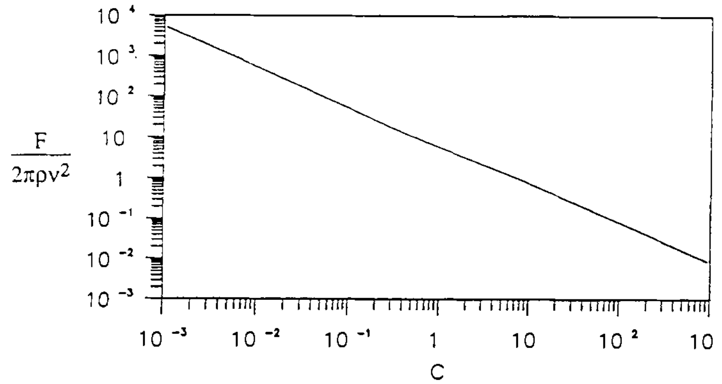

Fig. 1. The dimensionless force parameter $F/2\pi\rho\nu^2$ versus c.

where (23), (24), and (25) correspond to the Navier–Stokes equations in the r, θ, and ϕ directions, and the corresponding force and torque conditions are

$$\int_0^{\pi} [2a'a_0'\cot\theta + 3a_0'\cos\theta + aa_0' + a_0''\sin\theta + c_3\sin\theta\cos\theta]\,d\theta = 0\,, \tag{26}$$

$$\int_0^{\pi} (a'b_2 + a_0'b_1 + 3b_2\sin\theta)\sin\theta\,d\theta = \frac{M}{2\pi\rho\nu^2}\,. \tag{27}$$

Multiplying (25) by $\sin^2\theta$ and integrating it twice, we obtain the homogeneous solution

$$b_2 = \frac{B\sin\theta}{(c+1-\cos\theta)^2}\,, \tag{28}$$

which vanishes at $\theta = 0$ or π, where the azimuthal velocity must be continuous on the axis of symmetry. The value of the streamfunction is assumed to be zero on the axis of symmetry. The coefficients a_0 and c_3 are determined from Equations (23) and (24) with condition (26) and $a_0 = 0$ at $\theta = 0$ or π. Therefore,

$$a_0 = 0 \quad \text{and} \quad c_3 = 0\,. \tag{29}$$

By (27) and (28), we obtain

$$\frac{M}{2\pi\rho\nu^2} = B\left[-4 + 2(c+1)\ln\frac{c+2}{c} + \frac{8}{3c(c+2)}\right], \tag{30}$$

where B is a constant with a length unit. The dimensionless torque-to-force ratio M/FB, as obtained from (22) and (30), is plotted in Fig. 2.

4. Discussion

The flow field, produced by a point force and a point torque, can be represented by a round laminar jet as a first-order approximation in the far field with the effect of a torque being neglected. The second-order approximation yields a solution corresponding to a given torque

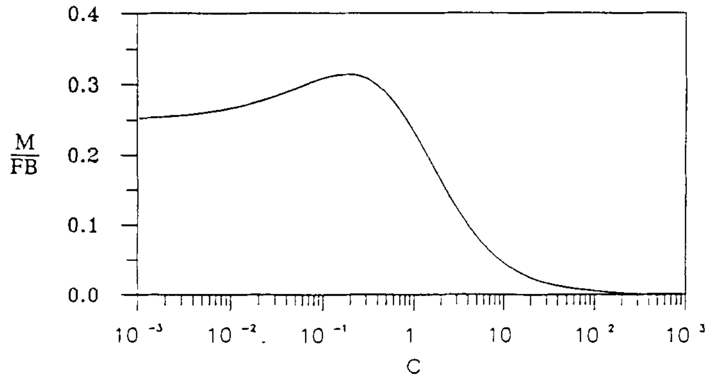

Fig. 2. The dimensionless torque-to-force ratio M/FB versus c.

and a zero force. The dimensionless parameter $F/(8\pi\rho\nu^2)$ may be interpreted as a Reynolds number. If $F/(8\pi\rho\nu^2)$ is large, the flow field can be interpreted as a strong jet. If $F/(8\pi\rho\nu^2)$ is small, the flow field can be interpreted as a weak jet.

For a strong jet, $c \ll 1$, the round-laminar-jet solution in the region where $\cos\theta$ is not close to 1, can be reduced to

$$a(\theta) = 2(1+\cos\theta)\left[1 - \frac{c}{1-\cos\theta} + O(c^2)\right], \tag{31a}$$

$$u(r,\theta) = -\frac{2\nu}{r} + O(c), \tag{31b}$$

$$v(r,\theta) = -\frac{2\nu}{r\sin\theta}(1+\cos\theta) + O(c), \tag{31c}$$

and the leading term of the swirling velocity is

$$b_2(\theta) = \frac{4M\sin\theta}{F(1-\cos\theta)^2}[1 + O(c)]. \tag{32}$$

Equation (32) shows that the swirling velocity makes no contribution to this flow region in the first-order approximation, since $F \sim O(1/c)$. The magnitude of the swirling velocity b_2 varies linearly with the ratio of the torque to the force in the second-order approximation. The leading term of (31a) is in agreement with Taylor's result by using potential-flow theory. From (31c), as $\theta \to 0$, the solution represents a line sink along the axis, $\theta = 0$, with a source density $-8\pi\nu$ per unit length. In the region where $\cos\theta \sim 1$, c and $1-\cos\theta$ are about the same order, and c can no longer be neglected. We introduce a new variable, $\xi = \sqrt{2/c}\tan\theta$, and $F \simeq 32\pi\rho\nu^2/3c$, then

$$a(\xi) = \frac{1}{1+\frac{1}{4}\xi^2}, \tag{33}$$

$$b_2(\xi) = \frac{\gamma\xi}{\left(1+\frac{1}{4}\xi^2\right)^2}, \tag{34}$$

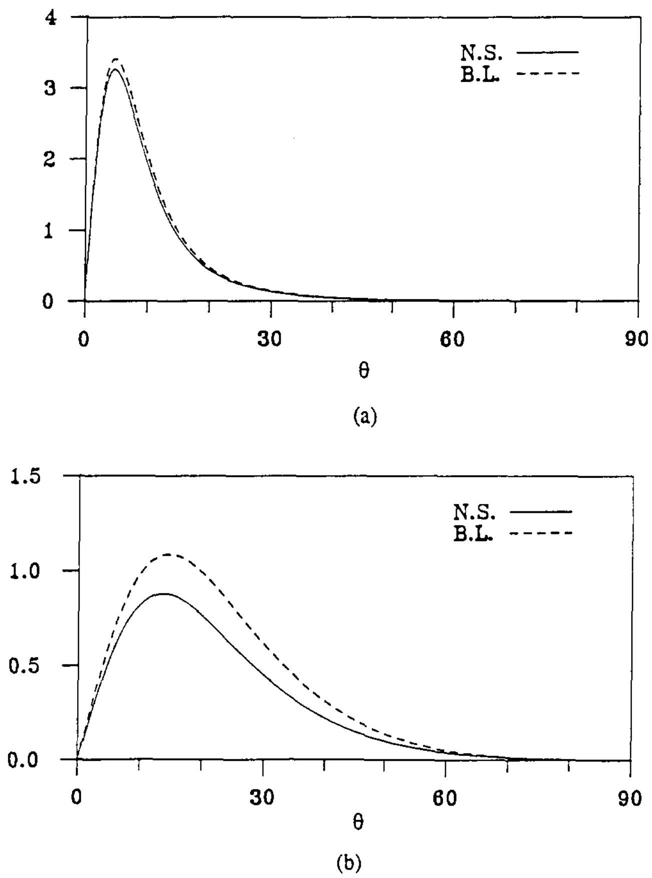

Fig. 3. Comparison of swirl velocity $2\pi\rho\nu x^2 w(\theta)/M$ from the boundary-layer equations and the Navier–Stokes equations with (a) $c = 0.01$ and (b) $c = 0.1$.

where

$$\gamma = \left(\frac{3F}{16\pi\rho\nu^2}\right)^{1/2}\left(\frac{3M}{16\pi\rho\nu^2}\right).$$

This result was first given by Loitsyanskii [15], who solved the boundary-layer equations in a cylindrical coordinate system (x, r, ϕ) with a strong-jet assumption. Equation (33) is identical to the result obtained by Schlichting [15] without swirl. In order to compare the swirling velocities obtained from the Navier–Stokes equations and the boundary-layer equations, both swirl velocities at a plane $x = \text{constant}$ are plotted in Fig. 3. We note from Fig. 3 that the swirling velocity obtained from the Navier–Stokes equations is always smaller than that obtained from the boundary-layer equations. As the value of c and B become small, which represents a strong jet with a weak swirling velocity, both results are very close to each other as shown in Fig. 3a. The leading term of the swirling velocity obtained from the boundary-layer theory in cylindrical coordinates is

$$w(r) = \frac{\nu}{x^2}\, b_2(r)\,. \tag{35}$$

For a weak jet, $c \gg 1$, a and b_2 can be approximated by

$$a(\theta) = \frac{F}{8\pi\rho\nu^2}\sin^2\theta\,, \tag{36}$$

$$b_2(\theta) = \frac{M}{8\pi\rho\nu^2}\sin\theta\,. \tag{37}$$

A weak jet corresponds to a low-Reynolds-number flow. The velocity field can be obtained by a linear superposition of one due to a point force and another due to a point torque. We note from (36) and (37) that $a(\theta)$ is related to F only, and $b_2(\theta)$ is related to M only. Actually, $a(\theta)$ corresponds to a Stokeslet which is the solution of a Stokes flow due to a point force, and $b_2(\theta)$ corresponds to a rotlet which is the solution of a Stokes flow due to a point torque (see Chwang and Wu [26]).

The solution given in (19) represents a jet of fluid moving away from the origin and entraining quiescent fluid from outside the jet. The edge of the jet can be defined as the place where the streamlines are at their minimum distances from the axis, and this edge occurs at $\theta = \theta_0$, where $\cos\theta_0 = 1/(1+c)$. If we assume that the maximum swirling velocity occurs at $\theta = \theta_m$, then by (28), we have

$$\cos\theta_m = \frac{4\cos\theta_0}{1+\sqrt{1+8\cos\theta_0}}\,. \tag{38}$$

From the above expression, θ_m is always less than θ_0 except at $\theta_0 = 0$ or $\pi/2$. This means that the maximum swirling speed occurs inside the jet except at zero Reynolds number or at an infinitely large Reynolds number where the maximum swirling speeds occur at the edge of the jet.

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